

A FAMILY OF QUADRATIC POLYNOMIAL DIFFERENTIAL SYSTEMS WITH ALGEBRAIC SOLUTIONS OF ARBITRARY HIGH DEGREE

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ABSTRACT. We show that the algebraic curve $a_0(x)(y - r(x)) + p_2(x)a'(x) = 0$, where $r(x)$ and $p_2(x)$ are polynomial of degree 1 and 2 respectively and $a_0(x)$ is a polynomial solution of the convenient Fucsh's equation, is an invariant curve of the quadratic planar differential system. We study the particular case when $a_0(x)$ is an orthogonal polynomials. We prove that that in this case the quadratic differential system is Liouvillian integrable.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Consider the set Σ of all planar real polynomial vector fields $\mathcal{X} = (P, Q)$ associated to the differential polynomial systems

$$(1) \quad \dot{x} = P(x, y), \quad \dot{y} = Q(x, y).$$

of degree $m = \max \{\deg P, \deg Q\}$, here the dot denotes derivative respect to time t .

Let U be an open and dense set in \mathbb{R}^2 . We say that a non-constant C^1 function $H: U \rightarrow \mathbb{R}$ is a *first integral* of the polynomial vector field \mathcal{X} on U , if $H(x(t), y(t))$ is constant for all values of t for which the solution $(x(t), y(t))$ of \mathcal{X} is defined on U . Clearly H is a first integral of \mathcal{X} on U if and only if $\mathcal{X}H = 0$ on U .

Let $\mathbb{C}[x, y]$ be the ring of all complex polynomials in the variables x and y , and let \mathcal{X} be a polynomial vector field of degree m , and let $g = g(x, y) \in \mathbb{C}[x, y]$. Then $g = 0$ is an *invariant algebraic curve* of \mathcal{X} if

$$(2) \quad \mathcal{X}g = P \frac{\partial g}{\partial x} + Q \frac{\partial g}{\partial y} = Kg,$$

where $K = K(x, y)$ is a polynomial of degree at most $m - 1$, which is called the *cofactor* of $g = 0$ (for more details see for instance [9]). If the polynomial g is irreducible in $\mathbb{C}[x, y]$, then we say that the invariant algebraic curve $g = 0$ is *irreducible* and that its *degree* is the degree of the polynomial g . We work with real polynomial vector fields but for these vector fields we consider complex invariant algebraic curves because sometimes the existence of a real first integral is forced by the existence of complex invariant algebraic curves, for more details on the so-called Darboux theory of integrability see for instance the Chapter 8 of [7].

A non-constant function $R: U \rightarrow \mathbb{R}$ is an *integrating factor* for the polynomial vector field \mathcal{X} , if one of the following three equivalent conditions holds

$$(3) \quad \frac{\partial(RP)}{\partial x} = -\frac{\partial(RQ)}{\partial y}, \quad \operatorname{div}(RP, RQ) = 0, \quad \mathcal{X}R = -R \operatorname{div}(P, Q)$$

on U . As usual the *divergence* of the vector field \mathcal{X} is defined by

$$\operatorname{div}(P, Q) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

Knowing an integrating factor R of the differential system (1) we can compute a first integral H of \mathcal{X} as follows

$$H = - \int R(x, y) P(x, y) dy + h(x),$$

where the function $h(x)$ is determined from the equality $\frac{\partial H}{\partial x} = R(x, y) Q(x, y)$.

Let $f_i, g_j, h_j \in \mathbb{C}[x, y]$ for $i = 1, \dots, p$ and $j = 1, \dots, q$. Then the (multi-valued) function

$$(4) \quad f_1^{\lambda_1} \dots f_p^{\lambda_p} e^{\mu_1 g_1 / h_1} \dots e^{\mu_q g_q / h_q}$$

with $\lambda_i, \mu_j \in \mathbb{C}$ is called a (*generalized*) *Darboux function*.

We say that a polynomial differential system (1) is *Liouvillian integrable* if it has a first integral or an integrating factor given by a generalized Darboux function, for more details see Singer [15].

The function $H = H(x, y)$ defined in an open subset \tilde{D}_1 of D such that its closure coincides with D is called a *first integral* if it is constant on the solutions of system (22) contained in \tilde{D}_1 , i.e. $\mathcal{X}(H)|_{\tilde{D}_1} = 0$.

From Jouanolou's Theorem (see for instance [8, 9]) it follows that for a given polynomial differential system of degree m the maximum *degree of its irreducible invariant algebraic curves* is bounded, since either it has a finite number $p < \frac{1}{2}m(m+1) + 2$, of invariant algebraic curves, or all its trajectories are contained in invariant algebraic curves and the system admits a *rational first integral*. Thus for each polynomial system there is a natural number N which bounds the degree of all its irreducible invariant algebraic curves. A natural question, going back to Poincaré (for more details see [14]), is to give an effective procedure to find N . Partial answer to this question were given in [2, 3, 4]. Of course, given such a bound, it is easy to compute the algebraic curves of the system.

Unfortunately, for the class of polynomial systems with fixed degree m , there does not, exist a uniform upper bound for N (see for instance [9]).

Another common suggestion is there is some number $M(m)$ for which all polynomials systems of degree m with invariant algebraic curves of degree greatest that $M(m)$ has a rational first integral (see for instance [6]). In [6] and [13] was proved that no such function $M(m)$ exist.

The aim of this paper, by developing the idea given in [6], is to give a family of polynomials systems of degree 2 without rational first integral but with irreducible invariant algebraic curves of arbitrary high degree.

The following results we can find in [10, 11]).

Proposition 1. *The most general differential system whit invariant algebraic curve $g = g(x, y) = 0$ is*

$$\dot{x} = -\nu \frac{\partial g}{\partial y} + \lambda_1 g, \quad \dot{y} = \nu \frac{\partial g}{\partial x} + \lambda_2 g,$$

where ν, λ_1 and λ_2 are arbitrary functions.

Theorem 2. *The algebraic curve $g = a_0(x)y + a_1(x) = 0$ is invariant for the quadratic system*

$$(5) \quad \begin{aligned} \dot{x} &= p_{22}x^2 + p_{21}x + p_{20} := p_2(x), \\ \dot{y} &= q_0y^2 + (q_{11}x + q_{10})y + q_{22}x^2 + q_{21}x + q_{20} := q_0y^2 + q_1(x)y + q_2(x), \end{aligned}$$

with cofactor $K = \alpha y + \beta x + \gamma$ if and only if

$$\alpha = q_0, \quad \alpha a_1(x) = p_2 a_0'(x) - ((\beta - q_{11})x + \gamma - q_{10}) a_0(x),$$

and $a_0 = a_0(x)$ is a polynomial solution of the Fucsh's equation

$$(6) \quad \begin{aligned} w'' + \frac{(2p_{22} + q_{11} - 2\beta)x + 2p_{21} + q_{10} - 2\gamma}{p_{22}x^2 + p_{21}x + p_{20}} w' - \frac{q_{11} - \beta}{p_{22}x^2 + p_{21}x + p_{20}} w \\ - \frac{(\beta^2 + \alpha q_{22} - \beta q_{11})x^2 + (\alpha q_{21} + 2\gamma\beta - q_{10}\beta - \gamma q_{11})x + \gamma^2 - \gamma q_{10} + \alpha q_{20}}{(p_{22}x^2 + p_{21}x + p_{20})^2} w = 0 \end{aligned}$$

Henceforth we shall consider that $q_0 = \alpha = 1$.

Our main results are the following

Theorem 3. *Under the assumptions of Theorem 2 we obtain that the most general quadratic polynomial differential system which admits the invariant algebraic curve*

$$(7) \quad a_0(y - ((\beta - q_{11})x + \gamma - q_{10})) + p_2 a_0' = 0,$$

where $a_0 = a_0(x)$ is a solution of the equation

$$(8) \quad p_2(x)a_0'' + r(x)a_0' + \kappa a_0 = 0.$$

where κ is a nonzero constant and $r(x) = (2p_{22} + q_{11} - \beta)x + \gamma + p_{21} - q_{10}$, is the system

$$(9) \quad \begin{aligned} \dot{x} &= p_{22}x^2 + p_{21}x + p_{20}, \\ \dot{y} &= q_0y^2 + (2\beta - 2p_{22} + \tau_{11})xy + (2\gamma + \tau_{10} - p_{21})y \\ &\quad + (2p_{22}^2 - (\tau_{11} - \tau_0 + 3\beta)p_{22} + \beta^2 + \beta\tau_{11})x^2 \\ &\quad + (2p_{22} - (2\beta - \tau_0 + \tau_{11})p_{21} + \gamma(\tau_{11} - 2p_{22} + 2\beta) + \beta\tau_{10})x \\ &\quad + p_{20}(2p_{22} - \tau_{11} + \tau_0 - \beta) + \gamma(\gamma + \tau_{10} - p_{21}), \end{aligned}$$

where τ_{11} , τ_{21} and τ_{20} are convenient constants.

The problem which appear is to deduce the conditions under which Fucsh equation (6) admits polynomial solutions. We shall study the subcase when the sought polynomials are hypergeometric functions (with convenient conditions on the parameters) and orthogonal polynomials.

A very important subcase of (8) is the hypergeometric differential equation

$$(10) \quad x(1-x)w'' + (c - (a+b+1)x)w' - abw = 0,$$

the solutions of this equation are the hypergeometric functions. We observe that the hypergeometric differential equation admits a polynomial solution if a or b are non-positive integer (for more details see [1]).

It is well known that orthogonal polynomials $f_0, f_1, \dots, f_n, \dots$ are the ones satisfying the differential equation (for more details see for instance [1])

$$(11) \quad \tau_2(x)f'' + \tau_1(x)f' + \tau_0f = 0,$$

where $\tau_j = \tau_j(x) = \sum_{n=1}^j \tau_{jn}x^n$ are a polynomials of degree at most j , for $j = 0, 1, 2$. The solution of (11) is an *orthogonal polynomial* if one of the following sets of conditions hold:

- (1) The polynomial $p_2(x)$ is quadratic with two distinct real roots, the root of polynomial $r = r(x)$ lies strictly between the roots of p_2 , and the leading terms of p_2 and r has the same sign. This case leads to the Jacobi-like polynomials which are solutions of the differential equation

$$(12) \quad (1 - x^2)f'' + (A - B - (A + B + 2)x)f' + n(n + A + B + 1)f = 0,$$

where A, B are real constants and n is natural number. Important special cases of Jacobi polynomials are Gegenbauer polynomials (with parameter $\gamma = A + 1/2$), Legendre polynomials ($A = B = 0$) and Chebyshev polynomials ($A = B = \pm 1/2$).

- (2) The polynomial $p_2(x)$ is linear. The roots of p_2 and r are different, and the leading terms of p_2 and r has the same sign if the root of r is less than the root of p_2 , or vice-versa. This case leads to the Laguerre-like polynomials which are solutions of the differential equation

$$(13) \quad xf'' + (A + 1 - x)f' + nf = 0,$$

where A is a real constants and n is natural number.

- (2) p_2 is just a nonzero constant. The leading term of r has the opposite sign of p_2 . This case leads to the Hermite-like polynomials (see for instance [1]) which are solutions of the differential equation

$$(14) \quad f'' - xf' + nf = 0,$$

where n is natural number.

Conjecture 4. *Differential system (9) admits a Liouvillian first integrals.*

Proposition 5. *Under the assumptions of Theorem 9 we obtain that the most general quadratic vector field which admits as a solution the curve (7) with a_0 the hypergeometric functions with a or b are non-positive integer, is*

$$(15) \quad \begin{aligned} \dot{x} &= x(1 - x), \\ \dot{y} &= y^2 + (2\beta - a - b + 1)xy + (c - 1 + 2\gamma)y + \gamma(\gamma + c - 1) \\ &\quad + (\beta^2 + (2 - a - b)\beta + (a - 1)(b - 1))x^2 \\ &\quad + (c\beta - ab + (1 - \gamma)(-2\beta - 1 + a + b))x \end{aligned}$$

Corollary 6. *The most general quadratic vector field which admits as a solution the curve (7) with a_0 the Jacobi polynomial is*

$$(16) \quad \begin{aligned} \dot{x} &= 1 - x^2, \\ \dot{y} &= y^2 + (2\beta - A - B)xy + (2\gamma + A - B)y \\ &\quad + ((\beta^2 + \beta - n(n + 1) - (1 + n + \beta)(A + B))x^2 \\ &\quad + ((\beta - \gamma)A - (\gamma + \beta)B + 2\gamma\beta)x \\ &\quad + (n + 1 - \gamma)B + (n + 1 + \gamma)A + n(n + 1) + \gamma^2 - \beta. \end{aligned}$$

The system (15) coincide with system (16) under the change

$$\begin{aligned} x &\longrightarrow \frac{x + 1}{2}, \quad y \longrightarrow \frac{y}{2}, \quad t \longrightarrow \frac{t}{2}, \quad \beta + 2\gamma \longrightarrow \gamma, \\ a &= -n, \quad b = 1 + n + A + B, \quad c = 1 + A. \end{aligned}$$

Corollary 7. *The most general quadratic vector field which admits as a solution the Laguerre polynomials is*

$$(17) \quad \begin{aligned} \dot{x} &= x, \\ \dot{y} &= y^2 + (2\beta - 1)xy + (2\gamma + A)y + (\beta - 1)\beta x^2 \\ &\quad + (\beta(A + 2\gamma - 1) + n + 1 - \gamma)x + A\gamma + \gamma^2. \end{aligned}$$

The system (15) coincide with system (17) under the change

$$\begin{aligned} x &\longrightarrow \frac{x}{b}, & \beta &\longrightarrow \beta b, \\ a &= -n, & c &= A + 1, \end{aligned}$$

and tend to infinity the parameter b .

Corollary 8. *The most general quadratic vector field which admits as a solution the curve (7) with a_0 the Hermite polynomial is*

$$(18) \quad \begin{aligned} \dot{x} &= 1, \\ \dot{y} &= y^2 + (2\beta - 1)xy + 2\gamma y \\ &\quad + (\beta^2 - 2\beta)x^2 + 2\gamma(\beta - 1)x + \gamma^2 - \beta + 2(n + 1) \end{aligned}$$

differential system (9) and its particular cases admits algebraic curves of arbitrary high degree.

Theorem 9. *Differential systems (15), (16), (17) and (18) are Liouvillian integrable.*

2. PROOF OF THE MAIN RESULTS

Proof of Theorem 3. In order to obtain a polynomial solutions of the equation (6) we consider the particular case when the quadratic polynomial $q_2 = q_2(x)$ is such that

$$(19) \quad q_2 = (\kappa + r')p_2 - (\beta x + \gamma)((\beta - q_{11})x + \gamma - q_{10}),$$

where κ is a constant.

Under this condition equation (6) takes the form (8). By compare equation (8) and (11) we obtain that

$$(20) \quad \tau_2(x) = p_2(x), \quad q_{11} = \tau_{11} - 2p_{22} + 2\beta, \quad q_{10} = \tau_{10} + 2\gamma - p_{21}, \quad \kappa = \tau_0$$

By solving equation (19) with respect to q_{22} , q_{21} and q_{20} and in view of (20) we deduce that

$$(21) \quad \begin{aligned} q_{22} &= 2p_{22}^2 - (\tau_{11} - \tau_0 + 3\beta)p_{22} + \beta^2 + \beta\tau_{11} \\ q_{21} &= 2p_{22} - (2\beta - \tau_0 + \tau_{11})p_{21} + \gamma(\tau_{11} - 2p_{22} + 2\beta) + \beta\tau_{10} \\ q_{20} &= p_{20}(2p_{22} - \tau_{11} + \tau_0 - \beta) + \gamma(\gamma + \tau_{10} - p_{21}), \end{aligned}$$

By inserting (21) into (5) we obtain (9). \square

Proof of Corollary 5. The hypergeometric functions are solutions of the differential equations (10). Thus by comparing with (8) we deduce that (we take $\alpha = 1$)

$$q_{11} = 2\beta - a - b + 1, \quad q_{10} = 2\gamma - 1 + c, \quad \kappa = -ab$$

consequently from (21) we give that

$$(22) \quad \begin{aligned} q_{20} &= \gamma(\gamma + c - 1), \\ q_{21} &= (1 - \gamma)(a + b - 2\beta - 1) + c\beta - ab, \\ q_{22} &= \beta^2 + (2 - a - b)\beta + (b - 1)(a - 1). \end{aligned}$$

Inserting into (9) we finally deduce (15). \square

Proof of Corollary 6. From (12) follows that $\tau_{11} = -A - B - 2$, $\tau_{10} = A - B$ and $\tau_0 = n(n + A + B + 1)$. Consequently, in view of (20) and (21) we obtain

$$\begin{aligned} p_{22} &= -1, & p_{21} &= 0, & p_{20} &= 1, \\ q_{11} &= 2\beta - A - B, & q_{10} &= 2\gamma + A - B, \\ q_{22} &= \beta^2 + \beta - n(n + 1) - (1 + n + \beta)(A + B), \\ q_{20} &= \gamma^2 - \beta + n(n + 1) + (1 + n + \gamma)A + (n + 1 - \gamma)B. \end{aligned}$$

Inserting into (9) we have the proof of the corollary. \square

Proof of Corollary 7. The Laguerre polynomials are solutions of the differential equations (12), thus from (8) we have that

$$q_{11} = -1 + 2\beta, \quad q_{10} = 1 + 2\gamma, \quad \kappa = -n$$

consequently in view of (22) we deduce that

$$q_{20} = \gamma + \gamma^2, \quad q_{21} = 2\gamma\beta + 1 - n - \gamma, \quad q_{22} = \beta^2 - \beta.$$

Inserting into (9) we finally deduce (17). \square

Proof of Corollary 8. The Hermite polynomials are solutions of the differential equations (12), thus from (8) we have that

$$q_{11} = 2\beta - 1, \quad q_{10} = 2\gamma, \quad \kappa = -\lambda$$

consequently in view of (22) we deduce that

$$q_{20} = \lambda - \beta + 1 + \gamma^2, \quad q_{21} = \gamma(2\beta - 1), \quad q_{22} = \beta^2 - \beta.$$

Inserting into (9) we finally deduce (17). \square

3. PROOF OF THEOREM 9

Proof of Theorem 9. After some computations it is possible to show that quadratic differential system (15) admits the following first integral

$$F = \frac{x^{1-c} g_1(x, y)}{g_2(x, y)},$$

where

$$\begin{aligned} g_1 &= (y + (1 - b + \beta)x + \gamma + c - 1 - a) F(1 + a + c, 1 + b - c, 2 - c; x) \\ &\quad + (1 + a - c)(1 - x) F(1 + a, b, c; x) + (1 + a - c), \\ g_2 &= (y + (1 - b + \beta)x + \gamma + c - 1 - a) F(a, b, c; x) \\ &\quad + a(1 - x) F(2 + a + c, 1 + b - c, 2 - c; x) + (1 + a - c) \end{aligned}$$

where $F(a, b, c; x) := {}_2F_1(a, b, c; x)$ is the hypergeometric function, where 2 refers to number of parameters in numerator and 1 refers to number of parameters in denominator.

In particular, differential system (15) for the values of the parameters

$$(23) \quad \beta = a + b - \frac{ab}{c} - 1, \quad \gamma = 1 - c,$$

takes the form

$$\begin{aligned} \dot{x} &= x(1 - x), \\ \dot{y} &= y^2 + (1 - c)y + \left(a + b - 1 - \frac{2ab}{c}\right)xy + \frac{ab(b - c)(a - c)}{c^2}x^2. \end{aligned}$$

In [6] was proved that this system which admits an algebraic curves of arbitrary high degree is not rational integrable but is Liouvillian integrable. Indeed, by considering that this system admits four algebraic invariant curves

$$\begin{aligned} g_1 &= x = 0, \quad g_2 = x - 1 = 0, \\ g_3 &= F_1\left(y - \frac{ab}{c}x\right) + x(1 - x)F_1' = 0 \\ g_4 &= F_2\left(y - \left(\frac{ab}{c} + 1 - c\right)x - c + 1\right) + x(1 - x)F_2' = 0, \end{aligned}$$

where $F_1 = F(a, b, c, x)$, $F_2 = F(1 + a - c, 1 + b - c, 2 - c, x)$ and a is negative integer. By considering that the cofactor are

$$K_1 = x - 1, \quad K_2 = x, \quad K_3 = y - \frac{(b-c)(a-c)}{c}x, \quad K_4 = y + \left(b + a - 1 - \frac{ba}{c}\right)x + 1 - c,$$

respectively, it is easy to obtain the existence of the Liouvillian first integral

$$F = \frac{x^{c-1}g_3}{g_4}$$

here c is a positive irrational number.

In view of the respectively corollary we have that the quadratic systems (16) and (17) are Liouvillian integrable.

The integrability of system (18) follows from the fact that this system can be rewritten as follows

$$\frac{dy}{dx} = y^2 + (2\beta - 1)xy + 2\gamma y + (\beta^2 - 2\beta)x^2 + 2\gamma(\beta - 1)x + \gamma^2 - \beta + 2(n + 1)$$

which is the Riccati equations with one solution

$$g_1 = (y + (\beta - 1)x + \gamma)H(x) + H'(x) = 0,$$

where H is the Hermite polynomial [5]. In short the theorem is proved. \square

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